Disproof of the List Hadwiger Conjecture*

János Barát[†] Gwenaël Joret[‡] David R. Wood[§]

October 12, 2011

Abstract

The List Hadwiger Conjecture asserts that every K_t-minor-free graph is t-choosable. We disprove this conjecture by constructing a K_{3t+2}-minor-free graph that is not $4t$-choosable for every integer $t \geq 1$.

1 Introduction

In 1943, Hadwiger [6] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory; see [26] for a survey[1].

Hadwiger Conjecture. *Every K_t-minor-free graph is $(t-1)$-colourable.*

The Hadwiger Conjecture holds for $t \leq 6$ (see [3, 6, 17, 18, 28]) and is open for $t \geq 7$. In fact, the following more general conjecture is open.

Weak Hadwiger Conjecture. *Every K_t-minor-free graph is ct-colourable for some constant $c \geq 1$.*

It is natural to consider analogous conjectures for list colourings[2]. First, consider the

*MSC: graph minors 05C83, graph coloring 05C15

[†]School of Mathematical Sciences, Monash University, VIC 3800, Australia (janos.barat@monash.edu). Research supported by OTKA Grant PD 75837, and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

[‡]Département d'Informatique, Université Libre de Bruxelles, Brussels, Belgium (gjoret@ulb.ac.be). Postdoctoral Researcher of the Fonds National de la Recherche Scientifique (F.R.S.–FNRS). Supported in part by the Actions de Recherche Concertées (ARC) fund of the Communauté française de Belgique. Also supported by an Endeavour Fellowship from the Australian Government.

[§]Department of Mathematics and Statistics, The University of Melbourne, Melbourne, Australia (woodd@unimelb.edu.au). Supported by a QEII Research Fellowship from the Australian Research Council.

[1]See [2] for undefined graph-theoretic terminology. Let $[a, b] := \{a, a + 1, \ldots, b\}$.

[2]A *list-assignment* of a graph G is a function L that assigns to each vertex v of G a set $L(v)$ of colours. G is *L-colourable* if there is a colouring of G such that the colour assigned to each vertex v is in $L(v)$. G is *k-choosable* if G is L-colourable for every list-assignment L with $|L(v)| \geq k$ for each vertex v of G. The *choice number* of G is the minimum integer k such that G is k-choosable. If G is k-choosable then G is also k-colourable—just use the same set of k colours for each vertex. Thus the choice number of G is at least the chromatic number of G. See [32] for a survey on list colouring.

choosability of planar graphs. Erdős et al. [5] conjectured that some planar graph is not 4-choosable, and that every planar graph is 5-choosable. The first conjecture was verified by Voigt [27] and the second by Thomassen [25]. Incidentally, Borowiecki [1] asked whether every K_t-minor-free graph is $(t-1)$-choosable, which is true for $t \leq 4$ but false for $t = 5$ by Voigt's example. The following natural conjecture arises (see [10, 30]).

List Hadwiger Conjecture. *Every K_t-minor-free graph is t-choosable.*

The List Hadwiger Conjecture holds for $t \leq 5$ (see [7, 20, 31]). Again the following more general conjecture is open.

Weak List Hadwiger Conjecture. *Every K_t-minor-free graph is ct-choosable for some constant $c \geq 1$.*

In this paper we disprove the List Hadwiger Conjecture for $t \geq 8$, and prove that $c \geq \frac{4}{3}$ in the Weak List Hadwiger Conjecture.

Theorem 1. *For every integer $t \geq 1$,*

(a) *there is a K_{3t+2}-minor-free graph that is not $4t$-choosable.*
(b) *there is a K_{3t+1}-minor-free graph that is not $(4t-2)$-choosable,*
(c) *there is a K_{3t}-minor-free graph that is not $(4t-3)$-choosable.*

Before proving Theorem 1, note that adding a dominant vertex to a graph does not necessarily increase the choice number (as it does for the chromatic number). For example, $K_{3,3}$ is 3-choosable but not 2-choosable. Adding one dominant vertex to $K_{3,3}$ gives $K_{1,3,3}$, which again is 3-choosable [16]. In fact, this property holds for infinitely many complete bipartite graphs [16]; also see [19].

2 Proof of Theorem 1

Let G_1 and G_2 be graphs, and let S_i be a k-clique in each G_i. Let G be a graph obtained from the disjoint union of G_1 and G_2 by pairing the vertices in S_1 and S_2 and identifying each pair. Then G is said to be obtained by *pasting* G_1 and G_2 on S_1 and S_2. The following lemma is well known.

Lemma 2. *Let G_1 and G_2 be K_t-minor-free graphs. Let S_i be a k-clique in each G_i. Let G be a pasting of G_1 and G_2 on S_1 and S_2. Then G is K_t-minor-free.*

Proof. Suppose on the contrary that K_{t+1} is a minor of G. Let $X_1, \dots, X_{t+1}$ be the corresponding branch sets. If some X_i does not intersect G_1 and some X_j does not intersect G_2, then no edge joins X_i and X_j, which is a contradiction. Thus, without loss of generality, each X_i intersects G_1. Let $X'_i := G_1[X_i]$. Since S_1 is a clique, X'_i is connected. Thus $X'_1, \dots, X'_{t+1}$ are the branch sets of a K_{t+1}-minor in G_1. This contradiction proves that G is K_t-minor-free. □

Let $K_{r\times 2}$ be the complete r-partite graph with r colour classes of size 2. Let $K_{1,r\times 2}$ be the complete $(r+1)$-partite graph with r colour classes of size 2 and one colour class of size 1. That is, $K_{r\times 2}$ and $K_{1,r\times 2}$ are respectively obtained from K_{2r} and K_{2r+1} by deleting a matching of r edges. The following lemma will be useful.

Lemma 3 ([8, 29])**.** *$K_{r\times 2}$ is $K_{\lfloor 3r/2\rfloor+1}$-minor-free, and $K_{1,r\times 2}$ is $K_{\lfloor 3r/2\rfloor+2}$-minor-free.*

Proof of Theorem 1. Our goal is to construct a K_p-minor-free graph and a non-achievable list assignment with q colours per vertex, where the integers p, q and r and a graph H are defined in the following table. Let $\{v_1w_1,\dots,v_rw_r\}$ be the deleted matching in H. By Lemma 3, the calculation in the table shows that H is K_p-minor-free.

case	p	q	r	H	
(a)	$3t+2$	$4t$	$2t+1$	$K_{r\times 2}$	$\lfloor \frac{3}{2}r\rfloor+1=3t+2=p$
(b)	$3t+1$	$4t-2$	$2t$	$K_{r\times 2}$	$\lfloor \frac{3}{2}r\rfloor+1=3t+1=p$
(c)	$3t$	$4t-3$	$2t-1$	$K_{1,r\times 2}$	$\lfloor \frac{3}{2}r\rfloor+2=3t=p$

For each vector $(c_1,\dots,c_r)\in[1,q]^r$, let $H(c_1,\dots,c_r)$ be a copy of H with the following list assignment. For each $i\in[1,r]$, let $L(w_i):=[1,q+1]\setminus\{c_i\}$. Let $L(u):=[1,q]$ for each remaining vertex u. There are $q+1$ colours in total, and $|V(H)|=q+2$. Thus in every L-colouring of H, two non-adjacent vertices receive the same colour. That is, $\mathrm{col}(v_i)=\mathrm{col}(w_i)$ for some $i\in[1,r]$. Since each $c_i\notin L(w_i)$, it is not the case that each vertex v_i is coloured c_i.

Let G be the graph obtained by pasting all the graphs $H(c_1,\dots,c_r)$, where $(c_1,\dots,c_r)\in[1,q]^r$, on the clique $\{v_1,\dots,v_r\}$. The list assignment L is well defined for G since $L(v_i)=[1,q]$. By Lemma 2, G is K_p-minor-free. Suppose that G is L-colourable. Let c_i be the colour assigned to each vertex v_i. Thus $c_i\in L(v_i)=[1,q]$. Hence, as proved above, the copy $H(c_1,\dots,c_r)$ is not L-colourable. This contradiction proves that G is not L-colourable. Each vertex of G has a list of q colours in L. Therefore G is not q-choosable. (It is easily seen that G is q-degenerate[3], implying G is $(q+1)$-choosable.) $\square$

Note that this proof was inspired by the construction of a non-4-choosable planar graph by Mirzakhani [15].

3 Conclusion

Theorem 1 disproves the List Hadwiger Conjecture. However, list colourings remain a viable approach for attacking Hadwiger's Conjecture. Indeed, list colourings provide potential routes around some of the known obstacles, such as large minimum degree, and lack of exact structure theorems; see [10, 11, 30, 31].

[3] A graph is d-degenerate if every subgraph has minimum degree at most d. Clearly every d-degenerate graph is $(d+1)$-choosable.

The following table gives the best known lower and upper bounds on the maximum choice number of K_t-minor-free graphs. Each lower bound is a special case of Theorem 1. Each upper bound (except $t = 5$) follows from the following degeneracy results. Every K_3-minor-free graph (that is, every forest) is 1-degenerate. Dirac [4] proved that every K_4-minor-free graph is 2-degenerate. Mader [14] proved that for $t \leq 7$, every K_t-minor-free graph is $(2t-5)$-degenerate. Jørgensen [9] and Song and Thomas [21] proved the same result for $t = 8$ and $t = 9$ respectively. Song [22] proved that every K_{10}-minor-free graph is 21-degenerate, and that every K_{11}-minor-free graph is 25-degenerate. In general, Kostochka [12, 13] and Thomason [23, 24] independently proved that every K_t-minor-free graph is $\mathcal{O}(t\sqrt{\log t})$-degenerate.

t	3	4	5	6	7	8	9	10	11	$\cdots$	t
lower bound	2	3	5	6	7	9	10	11	13	$\cdots$	$\frac{4}{3}t - c$
upper bound	2	3	5	8	10	12	14	22	26	$\cdots$	$\mathcal{O}(t\sqrt{\log t})$

The following immediate open problems arise:

- Is every K_6-minor-free graph 7-choosable?
- Is every K_6-minor-free graph 6-degenerate?
- Is every K_6-minor-free graph 6-choosable?

Acknowledgements

Thanks to Louis Esperet for stimulating discussions.

References

[1] Mieczyslaw Borowiecki. Research problem 172. *Discrete Math.*, 121:235–236, 1993. doi: 10.1016/0012-365X(93)90557-A.

[2] Reinhard Diestel. *Graph theory*, vol. 173 of *Graduate Texts in Mathematics*. Springer, 4th edn., 2010. http://diestel-graph-theory.com/. MR: 2744811.

[3] Gabriel A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952. doi: 10.1112/jlms/s1-27.1.85. MR: 0045371.

[4] Gabriel A. Dirac. Homomorphism theorems for graphs. *Math. Ann.*, 153:69–80, 1964. doi: 10.1007/BF01361708. MR: 0160203.

[5] Paul Erdős, Arthur L. Rubin, and Herbert Taylor. Choosability in graphs. In *Proc. West Coast Conference on Combinatorics, Graph Theory and Computing*, vol. XXVI of *Congress. Numer.*, pp. 125–157. Utilitas Math., 1980. http://www.renyi.hu/~p_erdos/1980-07.pdf. MR: 593902.

[6] Hugo Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943. MR: 0012237.

[7] Wenjie He, Wenjing Miao, and Yufa Shen. Another proof of the 5-choosability of K_5-minor-free graphs. *Discrete Math.*, 308(17):4024–4026, 2008. doi: 10.1016/j.disc.2007.07.089. MR: 2418108.

[8] Jaroslav Ivančo. Some results on the Hadwiger numbers of graphs. *Math. Slovaca*, 38(3):221–226, 1988. MR: 977899.

[9] Leif K. Jørgensen. Contractions to K_8. *J. Graph Theory*, 18(5):431–448, 1994. doi: 10.1002/jgt.3190180502. MR: 1283309.

[10] Ken-ichi Kawarabayashi and Bojan Mohar. A relaxed Hadwiger's conjecture for list colorings. *J. Combin. Theory Ser. B*, 97(4):647–651, 2007. doi: 10.1016/j.jctb.2006.11.002. MR: 2325803.

[11] Ken-ichi Kawarabayashi and Bruce Reed. Hadwiger's conjecture is decidable. In *Proc. 41st Annual ACM Symposium on Theory of Computing* (STOC '09), pp. 445–454. ACM, 2009. doi: 10.1145/1536414.1536476.

[12] Alexandr V. Kostochka. The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.*, 38:37–58, 1982. MR: 0713722.

[13] Alexandr V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984. doi: 10.1007/BF02579141. MR: 0779891.

[14] Wolfgang Mader. Homomorphiesätze für Graphen. *Math. Ann.*, 178:154–168, 1968. doi: 10.1007/BF01350657. MR: 0229550.

[15] Maryam Mirzakhani. A small non-4-choosable planar graph. *Bull. Inst. Combin. Appl.*, 17:15–18, 1996. MR: 1386951.

[16] Kyoji Ohba. Choice number of complete multipartite graphs with part size at most three. *Ars Combin.*, 72:133–139, 2004. MR: 2069052.

[17] Neil Robertson, Daniel P. Sanders, Paul D. Seymour, and Robin Thomas. The four-colour theorem. *J. Combin. Theory Ser. B*, 70(1):2–44, 1997. doi: 10.1006/jctb.1997.1750. MR: 1441258.

[18] Neil Robertson, Paul D. Seymour, and Robin Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica*, 13(3):279–361, 1993. doi: 10.1007/BF01202354. MR: 1238823.

[19] Yufa Shen, Wenjie He, Guoping Zheng, Yanning Wang, and Lingmin Zhang. On choosability of some complete multipartite graphs and Ohba's conjecture. *Discrete Math.*, 308(1):136–143, 2008. doi: 10.1016/j.disc.2007.03.059. MR: 2370527.

[20] Riste Škrekovski. Choosability of K_5-minor-free graphs. *Discrete Math.*, 190(1–3):223–226, 1998. doi: 10.1016/S0012-365X(98)00158-7. MR: 1639710.

[21] Zi-Xia Song and Robin Thomas. The extremal function for K_9 minors. *J. Combin. Theory Ser. B*, 96(2):240–252, 2006. doi: 10.1016/j.jctb.2005.07.008. MR: 2208353.

[22] Zixia Song. *Extremal Functions for Contractions of Graphs*. Ph.D. thesis, Georgia Institute of Technology, USA, 2004. CiteSeer: 10.1.1.92.1275. MR: 2706190.

[23] Andrew Thomason. An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.*, 95(2):261–265, 1984. doi: 10.1017/S0305004100061521. MR: 0735367.

[24] Andrew Thomason. The extremal function for complete minors. *J. Combin. Theory Ser. B*, 81(2):318–338, 2001. doi: 10.1006/jctb.2000.2013. MR: 1814910.

[25] Carsten Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62(1):180–181, 1994. doi: 10.1006/jctb.1994.1062. MR: 1290638.

[26] Bjarne Toft. A survey of Hadwiger's conjecture. *Congr. Numer.*, 115:249–283, 1996. MR: 1411244.

[27] Margit Voigt. List colourings of planar graphs. *Discrete Math.*, 120(1-3):215–219, 1993. doi: 10.1016/0012-365X(93)90579-I. MR: 1235909.

[28] Klaus Wagner. Über eine Eigenschaft der ebene Komplexe. *Math. Ann.*, 114:570–590, 1937. doi: 10.1007/BF01594196. MR: 1513158.

[29] David R. Wood. On the maximum number of cliques in a graph. *Graphs Combin.*, 23(3):337–352, 2007. doi: 10.1007/s00373-007-0738-8. MR: 2320588.

[30] David R. Wood. Contractibility and the Hadwiger conjecture. *European J. Combin.*, 31(8):2102–2109, 2010. doi: 10.1016/j.ejc.2010.05.015. MR: 2718284.

[31] David R. Wood and Svante Linusson. Thomassen's choosability argument revisited. *SIAM J. Discrete Math.*, 24(4):1632–1637, 2010. doi: 10.1137/100796649. MR: 2746713.

[32] Douglas R. Woodall. List colourings of graphs. In *Surveys in combinatorics*, vol. 288 of *London Math. Soc. Lecture Note Ser.*, pp. 269–301. Cambridge Univ. Press, 2001. MR: 1850711.